Algorithmic Problems on Temporal Graphs and a call for experiments

Paul G. Spirakis

Department of Computer Science,
University of Liverpool, UK and University of Patras , Greece

20th Symposium on Experimental Algorithms
SEA, 25 July 2022

Static and Temporal Graphs

Modern networks are highly dynamic:

- Social networks: friendships are added/removed, individuals leave, new ones enter
- Transportation networks: transportation units change with time their position in the network
- Physical systems: e.g. systems of interacting particles

The common characteristic in all these applications:

- the graph topology is subject to discrete changes over time

$\Rightarrow$ the notion of vertex adjacency must be appropriately re-defined (by introducing the time dimension in the graph definition)

Various graph concepts (e.g. reachability, connectivity):

- crucially depend on the exact temporal ordering of the edges

Temporal graphs (formally)

Definition (Temporal Graph)

A temporal graph is a pair (G, λ) where:

- $G = (V, E)$ is an underlying (di)graph and
- $\lambda : E \to 2^{\mathbb{N}}$ is a discrete time-labeling function.

- If $t \in \lambda(e)$ then edge e is available at time t

Temporal graphs (formally)

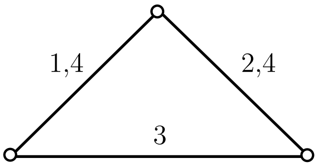
Definition (Temporal Graph)

A temporal graph is a pair (G, λ) where:

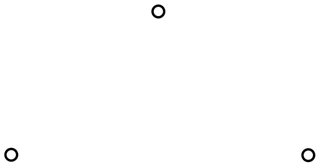
- $G = (V, E)$ is an underlying (di)graph and
- $\lambda : E \to 2^{\mathbb{N}}$ is a discrete time-labeling function.

- If $t \in \lambda(e)$ then edge e is available at time t

temporal graph:



temporal instances:



Temporal graphs (formally)

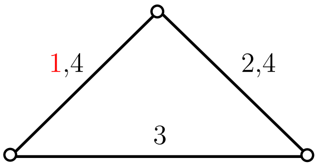
Definition (Temporal Graph)

A temporal graph is a pair (G, λ) where:

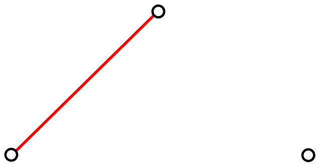
- $G = (V, E)$ is an underlying (di)graph and
- $\lambda : E \to 2^{\mathbb{N}}$ is a discrete time-labeling function.

- If $t \in \lambda(e)$ then edge e is available at time t

temporal graph:



temporal instances:



Temporal graphs (formally)

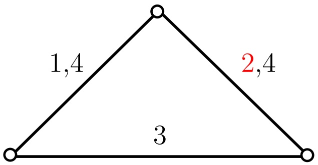
Definition (Temporal Graph)

A temporal graph is a pair (G, λ) where:

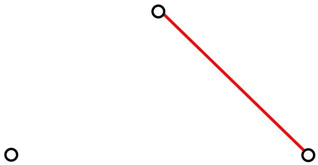
- $G = (V, E)$ is an underlying (di)graph and
- $\lambda : E \to 2^{\mathbb{N}}$ is a discrete time-labeling function.

- If $t \in \lambda(e)$ then edge e is available at time t

temporal graph:



temporal instances:



Temporal graphs (formally)

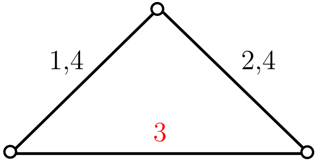
Definition (Temporal Graph)

A temporal graph is a pair (G, λ) where:

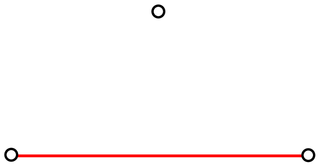
- $G = (V, E)$ is an underlying (di)graph and
- $\lambda : E \to 2^{\mathbb{N}}$ is a discrete time-labeling function.

- If $t \in \lambda(e)$ then edge e is available at time t

temporal graph:



temporal instances:



Temporal graphs (formally)

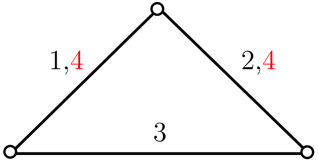
Definition (Temporal Graph)

A temporal graph is a pair (G, λ) where:

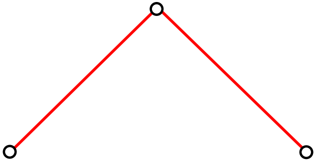
- $G = (V, E)$ is an underlying (di)graph and
- $\lambda : E \to 2^{\mathbb{N}}$ is a discrete time-labeling function.

- If $t \in \lambda(e)$ then edge e is available at time t

temporal graph:



temporal instances:



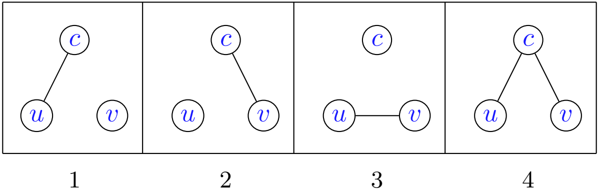
Temporal graphs (formally)

Definition (Temporal Graph)

A temporal graph is a pair (G, λ) where:

- $G = (V, E)$ is an underlying (di)graph and
- $\lambda : E \to 2^{\mathbb{N}}$ is a discrete time-labeling function.

Alternatively, we can view it as a sequence of static graphs, the snapshots:



Temporal graphs (formally)

Definition (Temporal Graph)

A temporal graph is a pair (G, λ) where:

- $G = (V, E)$ is an underlying (di)graph and
- $\lambda : E \to 2^{\mathbb{N}}$ is a discrete time-labeling function.

- Usually the input is a graph G with given labels $\{\lambda(e) : e \in E\}$
- Other models have been studied as well, e.g. various randomized models for temporal graphs:
  - every edge of G gets ρ random labels in each period α of time [Akrida, Gasieniec, Mertzios, Spirakis, *J. Par. Distr. Comp.*, 2016]
  - every edge appears according to a probability distribution [Akrida, Mertzios, Nikoletseas, Raptopoulos, Spirakis, Zamaraev, *J. Computer and System Sciences*, 2020]
  - random (temporal) edge permutation in an Erdös-Renyi random graph [Casteigts, Raskin, Renken, Zamaraev, *FOCS*, 2021]

Overview

- Temporal graphs
- Temporal parameters and temporal paths: a warm-up
- Temporal vertex cover
- Temporal transitive orientations
- Stochastic temporal graphs

Temporal paths

The conceptual shift from static to temporal graphs significantly impacts:

- the definition of basic graph parameters
- the type of tasks to be computed

Graph properties can be classified as:

- a-temporal, i.e. satisfied at every instance
  - connectivity at every point in time
- temporal, i.e. satisfied over time
  - communication routes over time

Temporal paths

The most natural known temporal notion in temporal graphs:

Definition (Temporal path; Time-respecting path; Journey)

Let (G, λ) be a temporal graph and $P = (e_1, e_2, \ldots, e_k)$ be a walk in G.
A temporal path is a sequence $((e_1, \ell_1), (e_2, \ell_2), \ldots, (e_k, \ell_k))$, where:

$$\ell_1 < \ell_2 < \ldots < \ell_k$$

and $\ell_i \in \lambda(e_i)$, $1 \leq i \leq k$.

Temporal paths

The most natural known temporal notion in temporal graphs:

Definition (Temporal path; Time-respecting path; Journey)

Let (G, λ) be a temporal graph and $P = (e_1, e_2, \ldots, e_k)$ be a walk in G.
A temporal path is a sequence $((e_1, \ell_1), (e_2, \ell_2), \ldots, (e_k, \ell_k))$, where:

$$\ell_1 < \ell_2 < \ldots < \ell_k$$

and $\ell_i \in \lambda(e_i)$, $1 \leq i \leq k$.

Motivation due to causality in information dissemination:

- information "flows" along edges whose labels respect time ordering

$\Rightarrow$ strictly increasing labels along the path

- a "static path" given "in pieces"

Temporal paths

The most natural known temporal notion in temporal graphs:

Definition (Temporal path; Time-respecting path; Journey)

Let (G, λ) be a temporal graph and $P = (e_1, e_2, \ldots, e_k)$ be a walk in G.
A temporal path is a sequence $((e_1, \ell_1), (e_2, \ell_2), \ldots, (e_k, \ell_k))$, where:

$$\ell_1 < \ell_2 < \ldots < \ell_k$$

and $\ell_i \in \lambda(e_i)$, $1 \leq i \leq k$.

Motivation due to causality in information dissemination:

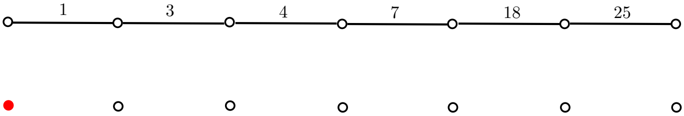
- information "flows" along edges whose labels respect time ordering

$\Rightarrow$ strictly increasing labels along the path

- a "static path" given "in pieces"
- A temporal path:

temporal path:

temporal instances:



Temporal paths

The most natural known temporal notion in temporal graphs:

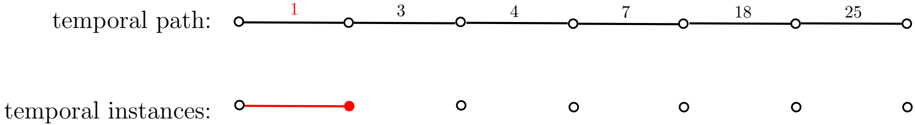
Definition (Temporal path; Time-respecting path; Journey)

Let (G, λ) be a temporal graph and $P = (e_1, e_2, \ldots, e_k)$ be a walk in G. A temporal path is a sequence $((e_1, \ell_1), (e_2, \ell_2), \ldots, (e_k, \ell_k))$, where:

$$\ell_1 < \ell_2 < \ldots < \ell_k$$

and $\ell_i \in \lambda(e_i)$, $1 \leq i \leq k$.

- A temporal path:



Temporal paths

The most natural known temporal notion in temporal graphs:

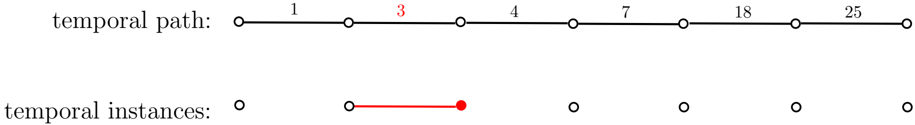
Definition (Temporal path; Time-respecting path; Journey)

Let (G, λ) be a temporal graph and $P = (e_1, e_2, \ldots, e_k)$ be a walk in G.
A temporal path is a sequence $((e_1, \ell_1), (e_2, \ell_2), \ldots, (e_k, \ell_k))$, where:

$$\ell_1 < \ell_2 < \ldots < \ell_k$$

and $\ell_i \in \lambda(e_i)$, $1 \leq i \leq k$.

- A temporal path:



Temporal paths

The most natural known temporal notion in temporal graphs:

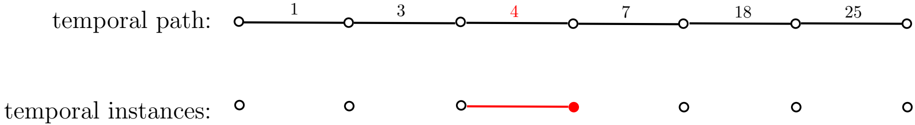
Definition (Temporal path; Time-respecting path; Journey)

Let (G, λ) be a temporal graph and $P = (e_1, e_2, \ldots, e_k)$ be a walk in G. A temporal path is a sequence $((e_1, \ell_1), (e_2, \ell_2), \ldots, (e_k, \ell_k))$, where:

$$\ell_1 < \ell_2 < \ldots < \ell_k$$

and $\ell_i \in \lambda(e_i)$, $1 \leq i \leq k$.

- A temporal path:



Temporal paths

The most natural known temporal notion in temporal graphs:

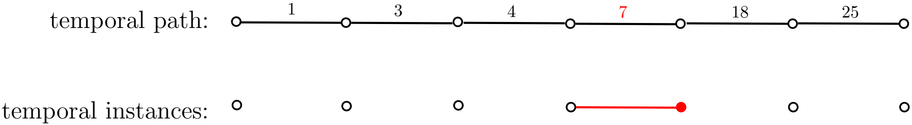
Definition (Temporal path; Time-respecting path; Journey)

Let (G, λ) be a temporal graph and $P = (e_1, e_2, \ldots, e_k)$ be a walk in G. A temporal path is a sequence $((e_1, \ell_1), (e_2, \ell_2), \ldots, (e_k, \ell_k))$, where:

$$\ell_1 < \ell_2 < \ldots < \ell_k$$

and $\ell_i \in \lambda(e_i)$, $1 \leq i \leq k$.

- A temporal path:



Temporal paths

The most natural known temporal notion in temporal graphs:

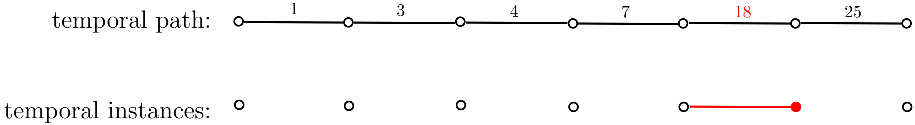
Definition (Temporal path; Time-respecting path; Journey)

Let (G, λ) be a temporal graph and $P = (e_1, e_2, \ldots, e_k)$ be a walk in G. A temporal path is a sequence $((e_1, \ell_1), (e_2, \ell_2), \ldots, (e_k, \ell_k))$, where:

$$\ell_1 < \ell_2 < \ldots < \ell_k$$

and $\ell_i \in \lambda(e_i)$, $1 \leq i \leq k$.

- A temporal path:



Temporal paths

The most natural known temporal notion in temporal graphs:

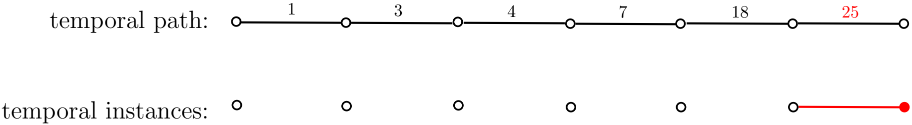
Definition (Temporal path; Time-respecting path; Journey)

Let (G, λ) be a temporal graph and $P = (e_1, e_2, \ldots, e_k)$ be a walk in G. A temporal path is a sequence $((e_1, \ell_1), (e_2, \ell_2), \ldots, (e_k, \ell_k))$, where:

$$\ell_1 < \ell_2 < \ldots < \ell_k$$

and $\ell_i \in \lambda(e_i)$, $1 \leq i \leq k$.

- A temporal path:



Temporal paths

The most natural known temporal notion in temporal graphs:

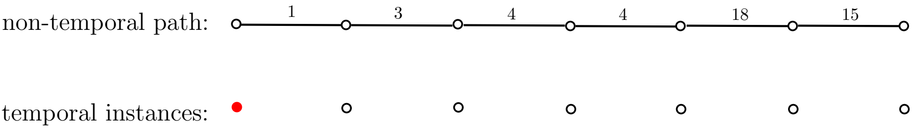
Definition (Temporal path; Time-respecting path; Journey)

Let (G, λ) be a temporal graph and $P = (e_1, e_2, \ldots, e_k)$ be a walk in G.
A temporal path is a sequence $((e_1, \ell_1), (e_2, \ell_2), \ldots, (e_k, \ell_k))$, where:

$$\ell_1 < \ell_2 < \ldots < \ell_k$$

and $\ell_i \in \lambda(e_i)$, $1 \leq i \leq k$.

- A non-temporal path:



Temporal paths

The most natural known temporal notion in temporal graphs:

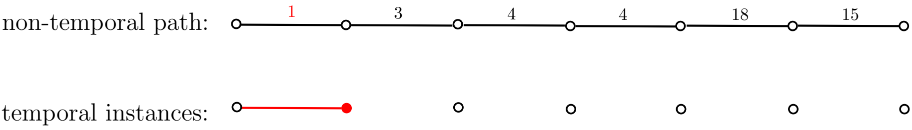
Definition (Temporal path; Time-respecting path; Journey)

Let (G, λ) be a temporal graph and $P = (e_1, e_2, \ldots, e_k)$ be a walk in G. A temporal path is a sequence $((e_1, \ell_1), (e_2, \ell_2), \ldots, (e_k, \ell_k))$, where:

$$\ell_1 < \ell_2 < \ldots < \ell_k$$

and $\ell_i \in \lambda(e_i)$, $1 \leq i \leq k$.

- A non-temporal path:



Temporal paths

The most natural known temporal notion in temporal graphs:

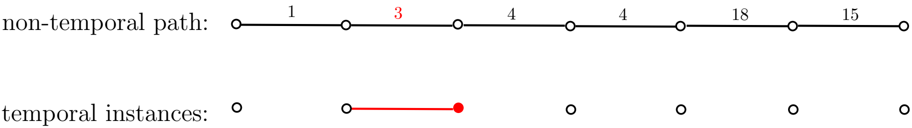
Definition (Temporal path; Time-respecting path; Journey)

Let (G, λ) be a temporal graph and $P = (e_1, e_2, \ldots, e_k)$ be a walk in G. A temporal path is a sequence $((e_1, \ell_1), (e_2, \ell_2), \ldots, (e_k, \ell_k))$, where:

$$\ell_1 < \ell_2 < \ldots < \ell_k$$

and $\ell_i \in \lambda(e_i)$, $1 \leq i \leq k$.

- A non-temporal path:



Temporal paths

The most natural known temporal notion in temporal graphs:

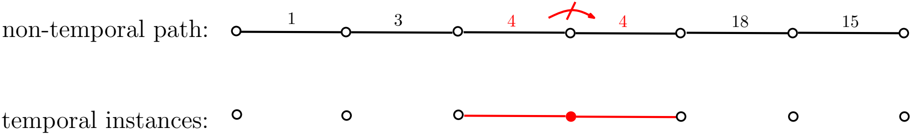
Definition (Temporal path; Time-respecting path; Journey)

Let (G, λ) be a temporal graph and $P = (e_1, e_2, \ldots, e_k)$ be a walk in G. A temporal path is a sequence $((e_1, \ell_1), (e_2, \ell_2), \ldots, (e_k, \ell_k))$, where:

$$\ell_1 < \ell_2 < \ldots < \ell_k$$

and $\ell_i \in \lambda(e_i)$, $1 \leq i \leq k$.

- A non-temporal path:



Temporal paths

The most natural known temporal notion in temporal graphs:

Definition (Temporal path; Time-respecting path; Journey)

Let (G, λ) be a temporal graph and $P = (e_1, e_2, \ldots, e_k)$ be a walk in G.
A temporal path is a sequence $((e_1, \ell_1), (e_2, \ell_2), \ldots, (e_k, \ell_k))$, where:

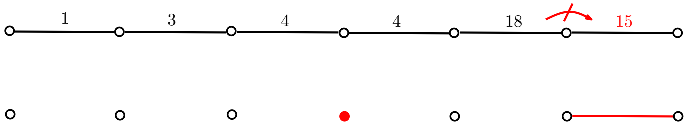
$$\ell_1 < \ell_2 < \ldots < \ell_k$$

and $\ell_i \in \lambda(e_i)$, $1 \leq i \leq k$.

- A non-temporal path:

non-temporal path:

temporal instances:



Temporal paths

The most natural known temporal notion in temporal graphs:

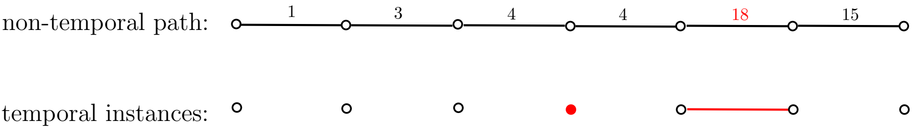
Definition (Temporal path; Time-respecting path; Journey)

Let (G, λ) be a temporal graph and $P = (e_1, e_2, \ldots, e_k)$ be a walk in G. A temporal path is a sequence $((e_1, \ell_1), (e_2, \ell_2), \ldots, (e_k, \ell_k))$, where:

$$\ell_1 < \ell_2 < \ldots < \ell_k$$

and $\ell_i \in \lambda(e_i)$, $1 \leq i \leq k$.

- A non-temporal path:



Temporal paths

The most natural known temporal notion in temporal graphs:

Definition (Temporal path; Time-respecting path; Journey)

Let (G, λ) be a temporal graph and $P = (e_1, e_2, \ldots, e_k)$ be a walk in G. A temporal path is a sequence $((e_1, \ell_1), (e_2, \ell_2), \ldots, (e_k, \ell_k))$, where:

$$\ell_1 < \ell_2 < \ldots < \ell_k$$

and $\ell_i \in \lambda(e_i)$, $1 \leq i \leq k$.

A temporal path can also be considered to be non-strict if we demand that:

$$\ell_1 \leq \ell_2 \leq \ldots \leq \ell_k$$

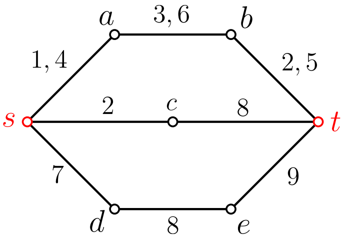
Metrics to optimize

Question: What is the temporal analogue of an s-t shortest path?

Answer: Not uniquely defined!

- topologically shortest path: smallest number of edges
- fastest path: smallest duration
- foremost path: smallest arrival time

Example:



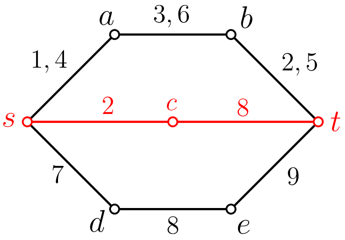
Metrics to optimize

Question: What is the temporal analogue of an s-t shortest path?

Answer: Not uniquely defined!

- topologically shortest path: smallest number of edges
- fastest path: smallest duration
- foremost path: smallest arrival time

Example:



shortest: s-c-t (two edges)

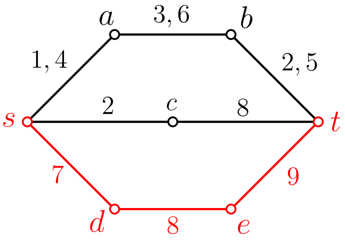
Metrics to optimize

Question: What is the temporal analogue of an s-t shortest path?

Answer: Not uniquely defined!

- topologically shortest path: smallest number of edges
- fastest path: smallest duration
- foremost path: smallest arrival time

Example:



shortest: s-c-t (two edges)

fastest: s-d-e-t (no intermediate waiting)

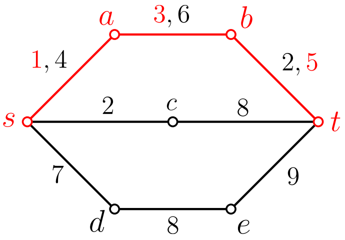
Metrics to optimize

Question: What is the temporal analogue of an s-t shortest path?

Answer: Not uniquely defined!

- topologically shortest path: smallest number of edges
- fastest path: smallest duration
- foremost path: smallest arrival time

Example:



shortest: s-c-t (two edges)

fastest: s-d-e-t (no intermediate waiting)

foremost: s-a-b-t (arriving at time 6)

Overview

- Temporal graphs
- Temporal paths: a warm-up
- Temporal vertex cover
- Temporal transitive orientations
- Stochastic temporal graphs

Basic definitions I

To specify a temporal graph class, we can:

- either restrict the underlying graph G,
- or restrict the labeling $\lambda : E \to 2^{\mathbb{N}}$ (or both)

Basic definitions I

To specify a temporal graph class, we can:

- either restrict the underlying graph G,
- or restrict the labeling $\lambda : E \to 2^{\mathbb{N}}$ (or both)

Definition (Temporal Graph Classes)

For a class $\mathcal{X}$ of static graphs we say that a temporal graph (G, λ) is

- $\mathcal{X}$ temporal, if $G \in \mathcal{X}$;
- always $\mathcal{X}$ temporal, if $G_i \in \mathcal{X}$ for every $i \in [T] = \{1, 2, \ldots, T\}$.

Basic definitions I

To specify a temporal graph class, we can:

- either restrict the underlying graph G,
- or restrict the labeling $\lambda : E \to 2^{\mathbb{N}}$ (or both)

Definition (Temporal Graph Classes)

For a class $\mathcal{X}$ of static graphs we say that a temporal graph (G, λ) is

- $\mathcal{X}$ temporal, if $G \in \mathcal{X}$;
- always $\mathcal{X}$ temporal, if $G_i \in \mathcal{X}$ for every $i \in [T] = \{1, 2, \ldots, T\}$.

Definition (Temporal Vertex Subset)

A pair $(u, t) \in V \times [T]$ is called the appearance of vertex u at time t.
A temporal vertex subset of (G, λ) is a set $\mathcal{S} \subseteq V \times [T]$ of vertex appearances in (G, λ).

Basic definitions II

Definition (Edge is Temporally Covered)

A vertex appearance (w, t) temporally covers an edge e if:

(i) w covers e, i.e. $w \in e$, and

(ii) $t \in \lambda(e)$, i.e. the edge e is active during the time slot t.

[Akrida, Mertzios, Spirakis, Zamaraev, *J. Comp. & System Sciences*, 2020]

Basic definitions II

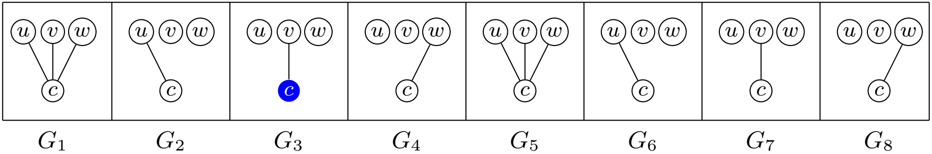
Definition (Edge is Temporally Covered)

A vertex appearance (w, t) temporally covers an edge e if:

(i) w covers e, i.e. $w \in e$, and

(ii) $t \in \lambda(e)$, i.e. the edge e is active during the time slot t.

Example:



- $(c, 3)$ temporally covers edge cv, but
- $(c, 3)$ temporally covers neither cu, nor cw.

[Akrida, Mertzios, Spirakis, Zamaraev, *J. Comp. & System Sciences*, 2020]

Basic definitions: Temporal Vertex Cover

Definition (Temporal Vertex Cover)

A temporal vertex cover of (G, λ) is a temporal vertex subset $\mathcal{S}$ of (G, λ) such that every edge $e \in E(G)$ is temporally covered by at least one vertex appearance in $\mathcal{S}$.

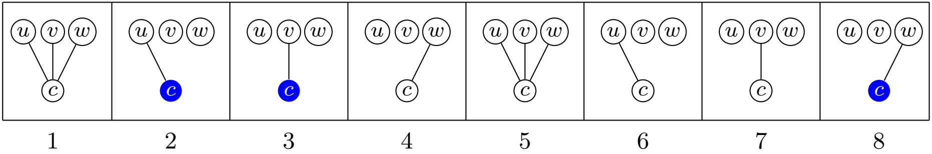
[Akrida, Mertzios, Spirakis, Zamaraev, *J. Comp. & System Sciences*, 2020]

Basic definitions: Temporal Vertex Cover

Definition (Temporal Vertex Cover)

A temporal vertex cover of (G, λ) is a temporal vertex subset $\mathcal{S}$ of (G, λ) such that every edge $e \in E(G)$ is temporally covered by at least one vertex appearance in $\mathcal{S}$.

Example



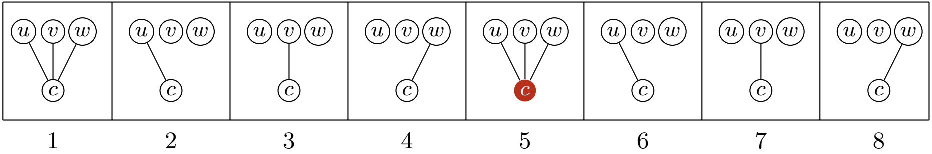
– $\{(c, 2), (c, 3), (c, 8)\}$ is a Temporal Vertex Cover

Basic definitions: Temporal Vertex Cover

Definition (Temporal Vertex Cover)

A temporal vertex cover of (G, λ) is a temporal vertex subset $\mathcal{S}$ of (G, λ) such that every edge $e \in E(G)$ is temporally covered by at least one vertex appearance in $\mathcal{S}$.

Example

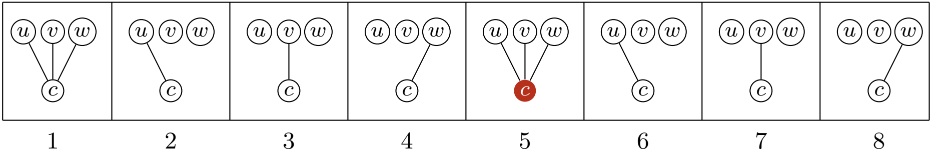


- $\{(c, 2), (c, 3), (c, 8)\}$ is a Temporal Vertex Cover
- $\{(c, 5)\}$ is a minimum Temporal Vertex Cover

Definition (Temporal Vertex Cover)

A temporal vertex cover of (G, λ) is a temporal vertex subset $\mathcal{S}$ of (G, λ) such that every edge $e \in E(G)$ is temporally covered by at least one vertex appearance in $\mathcal{S}$.

Example



TEMPORAL VERTEX COVER (TVC)

Input: A temporal graph (G, λ).

Output: A temporal vertex cover $\mathcal{S}$ of (G, λ) with the minimum $|\mathcal{S}|$.

Basic definitions: Sliding Window Temporal Vertex Cover

Definition (Time Windows)

1. For every time slot $t \in [1, T - \Delta + 1]$:
   the time window $W_t = [t, t + \Delta - 1]$ is the sequence of the Δ consecutive time slots $t, t + 1, \ldots, t + \Delta - 1$.

Basic definitions: Sliding Window Temporal Vertex Cover

Definition (Time Windows)

1. For every time slot $t \in [1, T - \Delta + 1]$:
   the time window $W_t = [t, t + \Delta - 1]$ is the sequence of the Δ consecutive time slots $t, t + 1, \ldots, t + \Delta - 1$.
2. $E[W_t] = \bigcup_{i \in W_t} E_i$ is the union of all edges appearing at least once in the time window W_t.
3. $\mathcal{S}[W_t] = \{(w, t) \in \mathcal{S} : t \in W_t\}$ is the restriction of the temporal vertex subset $\mathcal{S}$ to the window W_t.

Basic definitions: Sliding Window Temporal Vertex Cover

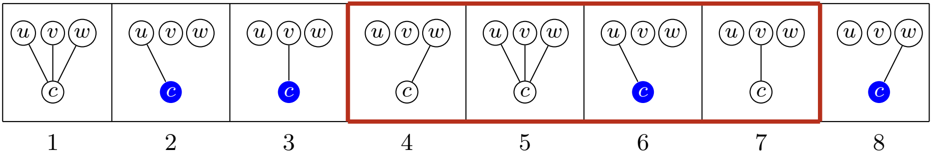
Definition (Sliding Δ-Window Temporal Vertex Cover)

A sliding Δ-window temporal vertex cover of (G, λ) is a temporal vertex subset $\mathcal{S}$ of (G, λ) such that:

- for every time window W_t and for every edge $e \in E[W_t]$,
- e is temporally covered by at least one vertex appearance $(w, t) \in \mathcal{S}[W_t]$.

Basic definitions: Sliding Window Temporal Vertex Cover

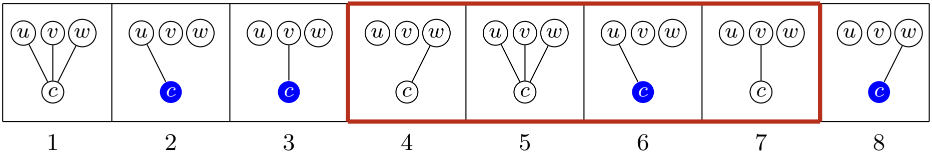
Example ($\Delta = 4$)



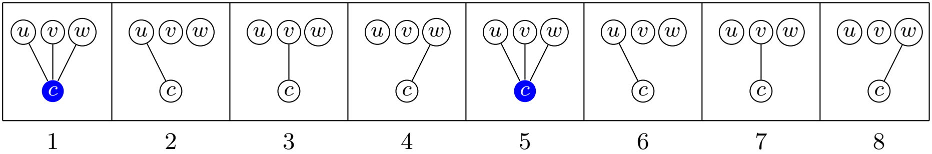
- $\{(c,2),(c,3),(c,6),(c,8)\}$ is not a sliding Δ-window temporal vertex cover, as edges $cv, cw \in E[W_4]$ are not temporally covered in window W_4.

Basic definitions: Sliding Window Temporal Vertex Cover

Example ($\Delta = 4$)



– $\{(c,2),(c,3),(c,6),(c,8)\}$ is not a sliding Δ-window temporal vertex cover, as edges $cv, cw \in E[W_4]$ are not temporally covered in window W_4.



– $\{(c,1),(c,5)\}$ is a sliding Δ-window temporal vertex cover.

Basic definitions: Sliding Window Temporal Vertex Cover

SLIDING WINDOW TEMPORAL VERTEX COVER (SW-TVC)

Input: A temporal graph (G, λ) with lifetime T, and an integer $\Delta \leq T$.
Output: A sliding Δ-window temporal vertex cover $\mathcal{S}$ of (G, λ) with the minimum $|\mathcal{S}|$.

Motivation:

- **(static) Vertex Cover:**
  network surveillance (e.g. CCTV cameras etc.)
- **Temporal Vertex Cover:**
  network surveillance in a dynamic network
- **Sliding Window Temporal Vertex Cover:**
  dynamic surveillance in every possible Δ-time window
  (e.g. for crimes that need time Δ to be performed)

Temporal Vertex Cover: the star temporal case

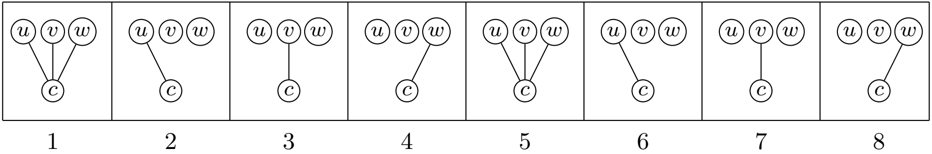
Lemma (Akrida et al., *J. Comp. & System Sciences*, 2020)

TVC on star temporal graphs is equivalent to SET COVER.

- leafs of the underlying star $\leftrightarrow$ ground set of the SET COVER instance
- each snapshot graph $\leftrightarrow$ a set in the SET COVER instance

Goal: Choose sets (snapshots) to cover all elements (leafs' edges)

Example:



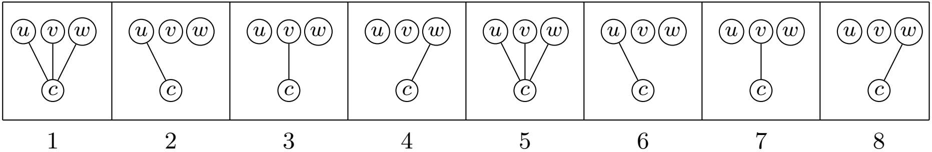
Lemma (Akrida et al., *J. Comp. & System Sciences*, 2020)

TVC *on star temporal graphs is equivalent to* SET COVER.

- leafs of the underlying star $\leftrightarrow$ ground set of the SET COVER instance
- each snapshot graph $\leftrightarrow$ a set in the SET COVER instance

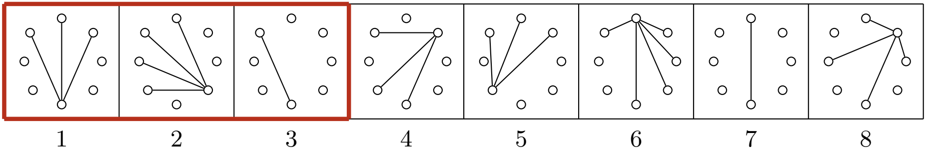
Goal: Choose sets (snapshots) to cover all elements (leafs' edges)

Example:

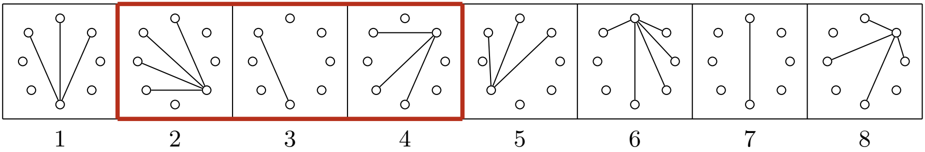


1. Universe: $\{u, v, w\}$
2. Sets: $S_1 = \{u, v, w\}$, $S_2 = \{u\}$, $S_3 = \{v\}$, $S_4 = \{w\}$, ...

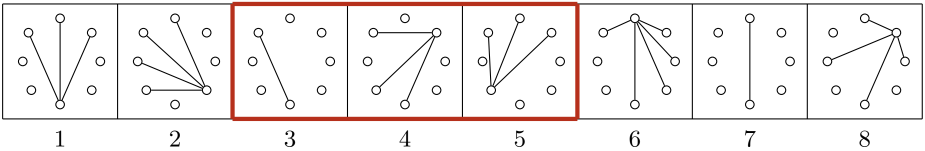
SW-TVC: **always star** temporal graphs



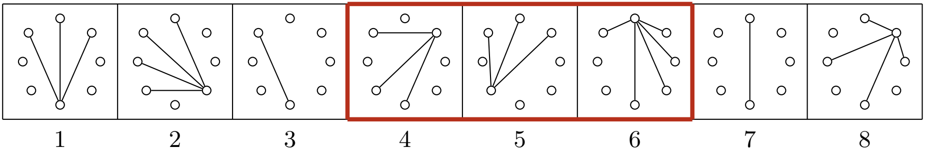
SW-TVC: **always star** temporal graphs



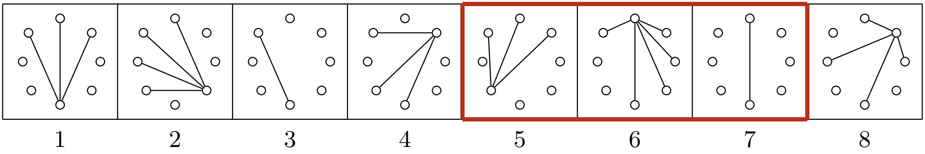
SW-TVC: **always star** temporal graphs



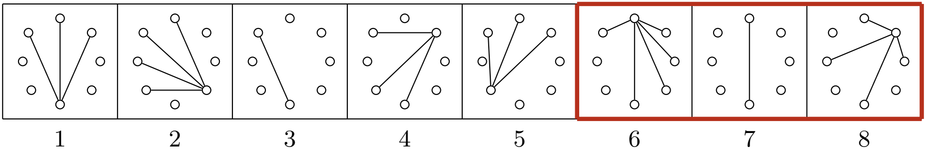
SW-TVC: **always star** temporal graphs



SW-TVC: **always star** temporal graphs



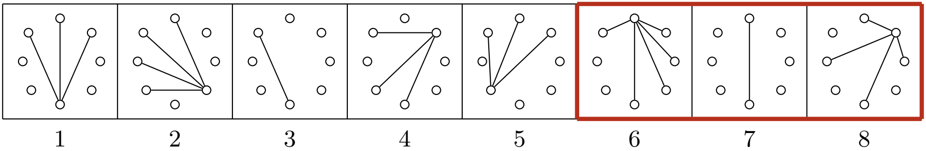
SW-TVC: **always star** temporal graphs



- On always star temporal graphs, a minimum size SW-TVC contains at most one vertex (the star center) in each snapshot

$\Rightarrow$ we assign a Boolean variable $x_i \in \{0, 1\}$ for the snapshot at time i

SW-TVC: **always star** temporal graphs

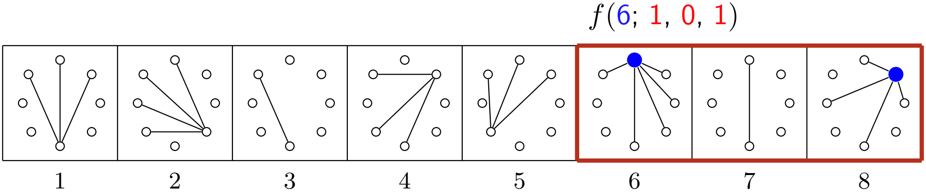


- On always star temporal graphs, a minimum size SW-TVC contains at most one vertex (the star center) in each snapshot

$\Rightarrow$ we assign a Boolean variable $x_i \in \{0, 1\}$ for the snapshot at time i

- For variables $x_1, x_2, \ldots, x_\Delta$ we define $f(t; x_1, x_2, \ldots, x_\Delta)$ to be the smallest cardinality of a sliding Δ-window temporal vertex cover $\mathcal{S}$ of $(G, \lambda)|_{[1,t+\Delta-1]}$, such that the solution in the time window $W_t = \{t, \ldots, t + \Delta - 1\}$ is given by the variables $x_1, x_2, \ldots, x_\Delta$.

SW-TVC: **always star** temporal graphs



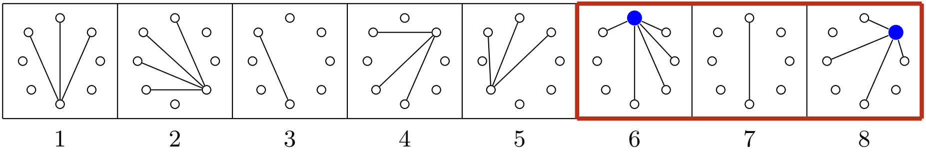
- On always star temporal graphs, a minimum size SW-TVC contains at most one vertex (the star center) in each snapshot

$\Rightarrow$ we assign a Boolean variable $x_i \in \{0, 1\}$ for the snapshot at time i

- For variables $x_1, x_2, \ldots, x_\Delta$ we define $f(t; x_1, x_2, \ldots, x_\Delta)$ to be the smallest cardinality of a sliding Δ-window temporal vertex cover $\mathcal{S}$ of $(G, \lambda)|_{[1,t+\Delta-1]}$, such that the solution in the time window $W_t = \{t, \ldots, t + \Delta - 1\}$ is given by the variables $x_1, x_2, \ldots, x_\Delta$.

SW-TVC: **always star** temporal graphs

$f(6; 1, 0, 1)$



- On always star temporal graphs, a minimum size SW-TVC contains at most one vertex (the star center) in each snapshot

$\Rightarrow$ we assign a Boolean variable $x_i \in \{0, 1\}$ for the snapshot at time i

- For variables $x_1, x_2, \ldots, x_\Delta$ we define $f(t; x_1, x_2, \ldots, x_\Delta)$ to be the smallest cardinality of a sliding Δ-window temporal vertex cover $\mathcal{S}$ of $(G, \lambda)|_{[1,t+\Delta-1]}$, such that the solution in the time window $W_t = \{t, \ldots, t + \Delta - 1\}$ is given by the variables $x_1, x_2, \ldots, x_\Delta$.

Lemma (dynamic programming)

$$f(t; x_1, x_2, \ldots, x_\Delta) = x_\Delta + \min_{y \in \{0,1\}} \{f(t-1; y, x_1, x_2, \ldots, x_{\Delta-1})\}$$

Theorem (always star temporal graphs)

SW-TVC *on always star temporal graphs can be solved in* $O(T\Delta(n+m) \cdot 2^{\Delta})$ *time.*

SW-TVC

Theorem (always star temporal graphs)

SW-TVC *on always star temporal graphs can be solved in* $O(T\Delta(n+m)\cdot 2^{\Delta})$ *time.*

Theorem (the general case)

SW-TVC *on general temporal graphs can be solved in* $O(T\Delta(n+m)\cdot 2^{n(\Delta+1)})$ *time.*

Main idea:

- for each of the Δ snapshots in the (currently) last Δ-window, we enumerate all 2^n vertex subsets,
- instead of just enumerating over the truth values of Δ Boolean variables ("always star" case)

SW-TVC

Theorem (always star temporal graphs)

SW-TVC *on always star temporal graphs can be solved in* $O(T\Delta(n+m) \cdot 2^{\Delta})$ *time.*

Theorem (the general case)

SW-TVC *on general temporal graphs can be solved in* $O(T\Delta(n+m) \cdot 2^{n(\Delta+1)})$ *time.*

We can prove:

Corollary

Our $O(T\Delta(n+m) \cdot 2^{n(\Delta+1)})$*-time algorithm is asymptotically almost optimal (assuming ETH).*

Δ-TVC

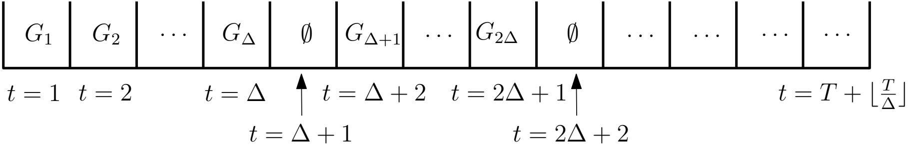
If the parameter Δ (the size of a sliding window) is fixed, we refer to SW-TVC as Δ-TVC (i.e. Δ is a part of the problem name).

Δ-TVC

If the parameter Δ (the size of a sliding window) is fixed, we refer to $\mathrm{SW\text{-}TVC}$ as $\Delta\text{-}\mathrm{TVC}$ (i.e. Δ is a part of the problem name).

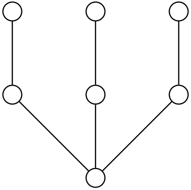
Observation

$(\Delta+1)\text{-}\mathrm{TVC}$ *is at least as hard as* $\Delta\text{-}\mathrm{TVC}$.



2-TVC for $\max \deg \leq 3$

Let $\mathcal{X}$ be the class of graphs whose connected components are induced subgraphs of graph Ψ, with maximum degree 3:



Ψ

Clearly, VERTEX COVER is linearly solvable on graphs from $\mathcal{X}$.

Theorem (Akrida et al., *J. Comp. & System Sciences*, 2020)

There is no PTAS for 2-TVC on always $\mathcal{X}$ temporal graphs.

- What is the complexity for (always) maximum degree 2 ?

2-TVC for $\max \deg \leq 2$

Our results when the underlying graph is a path or a cycle:

- linear-time algorithm for TVC (no sliding windows)
- $2\text{-}\mathrm{TVC}$ is NP-hard
- PTAS for $\Delta\text{-}\mathrm{TVC}$, for any $\Delta \geq 2$

[Hamm, Klobas, Mertzios, Spirakis, *AAAI*, 2022]

2-TVC for $\max \deg \leq 2$

Greedy linear-time algorithm for TVC on paths:

- visit the vertices from left to right

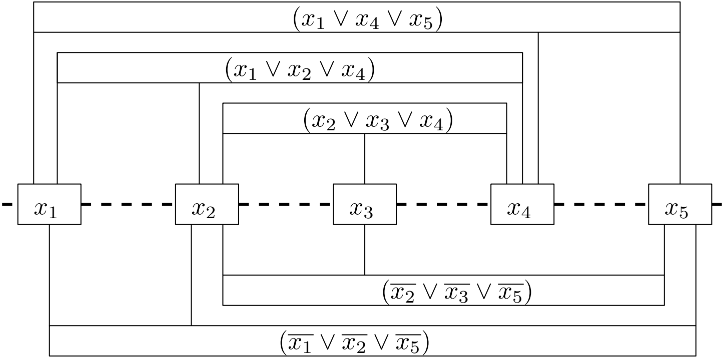
for every $i = 1, 2, \ldots, n-1$ **do**
  if e_i and e_{i+1} appear* at the same time t (for some t) **then**
    add (v_{i+1}, t) to $\mathcal{C}$ (where $e_i \cap c_{i+1} = \{v_{i+1}\}$)
    $i = i + 2$
  else
    Add to $\mathcal{C}$ an arbitrary (v_i, t) or (v_{i+1}, t), where $t \in \lambda(e_i)$.
    $i = i + 1$
return $\mathcal{C}$.

* Denote by $e_i = v_i v_{i+1}$, for every $i = 1, 2, \ldots, n-1$.

2-TVC is NP-hard on temporal paths

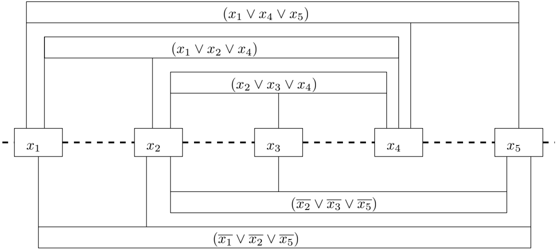
Reduction from planar monotone rectilinear 3SAT.

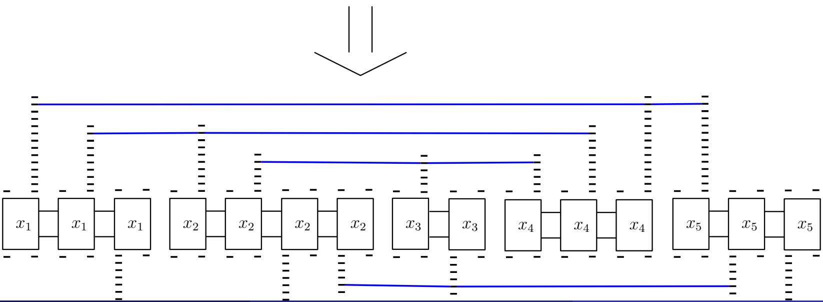
$$\phi = (x_2 \vee x_3 \vee x_4) \wedge (x_1 \vee x_2 \vee x_4) \wedge (x_1 \vee x_4 \vee x_5) \wedge (\overline{x_2} \vee \overline{x_3} \vee \overline{x_5}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_5})$$



2-TVC is NP-hard on temporal paths

High-level construction:





PTAS for Δ-TVC on temporal paths

Reduction to this problem:

GEOMETRIC HITTING SET

Input: A pair $R = (P, D)$ (range space), where P is a set of points in $\mathbb{R}^2$ and D is a set of regions covering all points of P.
Output: A smallest subset of points $S \subseteq P$, such that every region in D contains at least one point of S.

PTAS for Δ-TVC on temporal paths

Reduction to this problem:

GEOMETRIC HITTING SET

Input: A pair $R = (P, D)$ (range space), where P is a set of points in $\mathbb{R}^2$ and D is a set of regions covering all points of P.
Output: A smallest subset of points $S \subseteq P$, such that every region in D contains at least one point of S.

PTAS for r-admissible set regions:

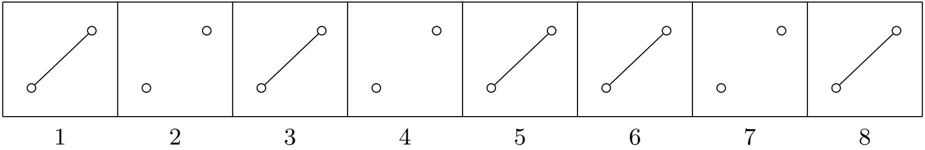
- boundaries of $s_1, s_2 \in D$ intersect at most r times
- $s_1 \setminus s_2$ and $s_2 \setminus s_1$ are connected regions

Theorem (Mustafa and Ray, *Discrete and Computat. Geometry*, 2010)

For every $\varepsilon > 0$, there is an $(1 + \varepsilon)$-approximation algorithm for GEOMETRIC HITTING SET *that runs in $O(|D||P|^{O(\varepsilon^{-2})})$ time.*

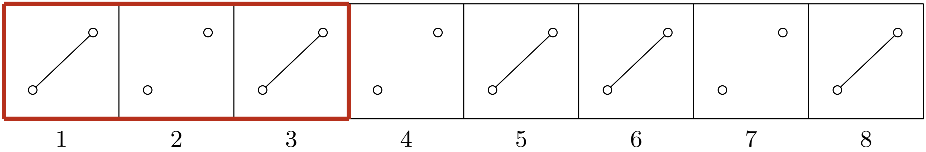
SW-TVC: approximation algorithms II

Single-edge temporal graph: **exact algorithm**



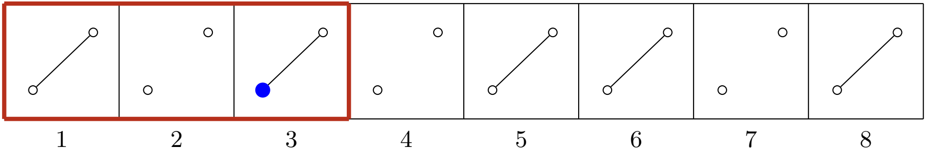
SW-TVC: approximation algorithms II

Single-edge temporal graph: **exact algorithm**



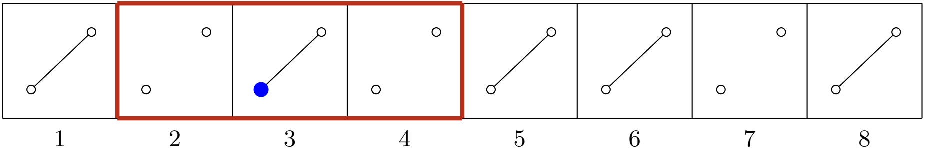
SW-TVC: approximation algorithms II

Single-edge temporal graph: **exact algorithm**



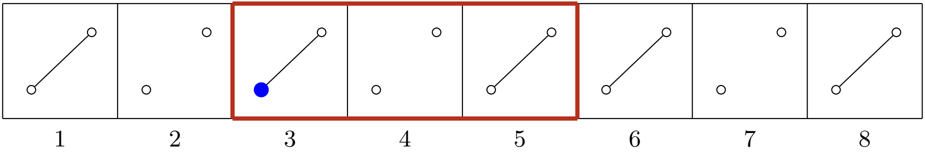
SW-TVC: approximation algorithms II

Single-edge temporal graph: **exact algorithm**



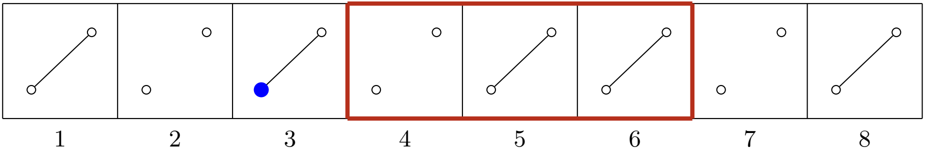
SW-TVC: approximation algorithms II

Single-edge temporal graph: **exact algorithm**



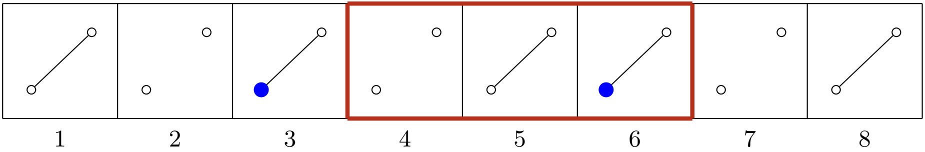
SW-TVC: approximation algorithms II

Single-edge temporal graph: exact algorithm



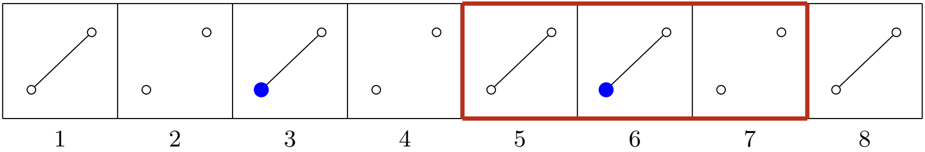
SW-TVC: approximation algorithms II

Single-edge temporal graph: exact algorithm



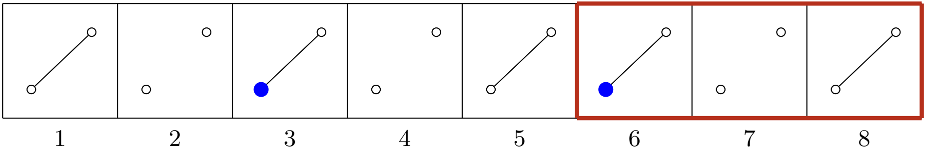
SW-TVC: approximation algorithms II

Single-edge temporal graph: **exact algorithm**



SW-TVC: approximation algorithms II

Single-edge temporal graph: exact algorithm



SW-TVC: approximation algorithms II

Single-edge temporal graph: exact algorithm

1. In the first window $W_t = [1, \Delta]$: cover the edge at the latest time slot it appears
   (to "cover" as many other windows as possible)
2. Remove all windows that are now covered
3. Repeat

- greedy algorithm
- linear time

SW-TVC: approximation algorithms II

Always degree at most d temp. graphs: d-approx. algorithm

Main idea:

- solve independently each single-edge subgraph of G
- take the union of the solutions

SW-TVC: approximation algorithms II

Always degree at most d temp. graphs: **d-approx. algorithm**

Main idea:

- solve independently each single-edge subgraph of G
- take the union of the solutions

Lemma (Akrida et al., *J. Comp. & System Sciences*, 2020)

There is a $O(mT)$-time d-approximation algorithm for $\mathrm{SW\text{-}TVC}$ on always degree at most d temporal graphs.

- Can we do better?

SW-TVC: approximation algorithms II

Always degree at most d temp. graphs:
$(d-1)$**-approx. algorithm**

Main idea:

- instead of single edges, solve first $\mathrm{SW\text{-}TVC}$ independently every possible P_3 in (G, λ)
- take the union of the solutions

$\mathrm{SW\text{-}TVC}$: approximation algorithms II

Always degree at most d temp. graphs:
$(d-1)$-approx. algorithm

Main idea:

- instead of single edges, solve first $\mathrm{SW\text{-}TVC}$ independently every possible P_3 in (G, λ)
- take the union of the solutions

Lemma (Hamm, Klobas, Mertzios, Spirakis, *AAAI*, 2022)

There is a $O\left(m^2T^2\right)$-time $(d-1)$-approximation algorithm for $\mathrm{SW\text{-}TVC}$ on always degree at most d temporal graphs.

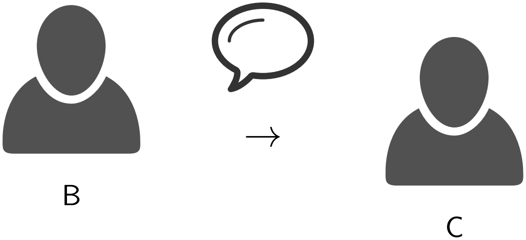
- We suspect an approximation ratio $c \cdot d$ ...

Overview

- Temporal graphs
- Temporal paths: a warm-up
- Temporal vertex cover
- Temporal transitive orientations
- Stochastic temporal graphs

Temporal transitive orientation

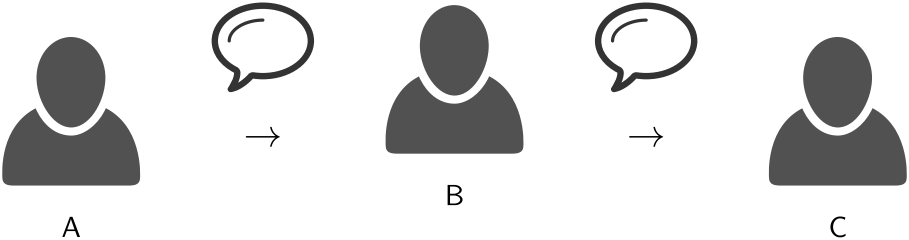
Motivation: Rumor Spreading



Scenario: C hears a rumor from B, asks for the source A, then (later) confirms with A whether the rumor is true.

Temporal transitive orientation

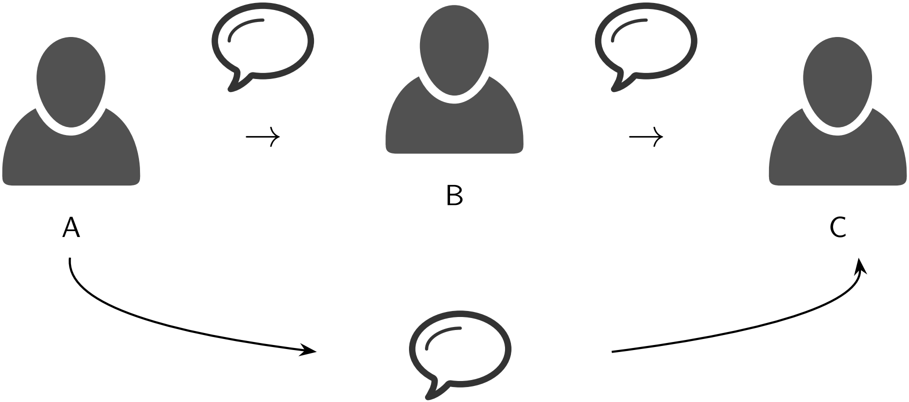
Motivation: Rumor Spreading



Scenario: C hears a rumor from B, asks for the source A, then (later) confirms with A whether the rumor is true.

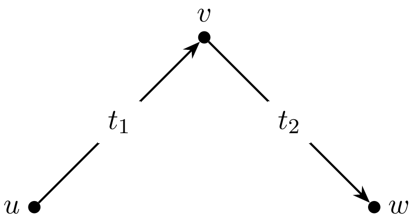
Temporal transitive orientation

Motivation: Rumor Spreading



Scenario: C hears a rumor from B, asks for the source A, then (later) confirms with A whether the rumor is true.

Temporal Transitivity

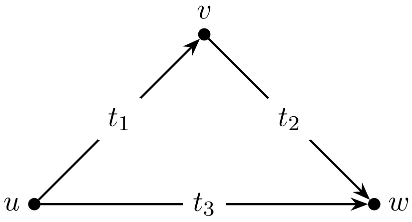


Temporal Transitivity

If (uv, t_1) and (vw, t_2) are temporal edges with $t_1 \leq t_2$,

[Mertzios, Molter, Renken, Spirakis, Zschoche, *MFCS*, 2021]

Temporal Transitivity

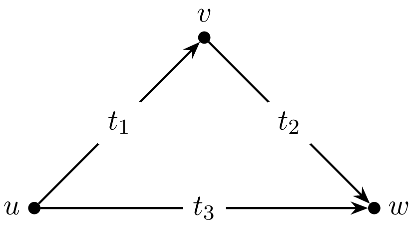


Temporal Transitivity

If (uv, t_1) and (vw, t_2) are temporal edges with $t_1 \leq t_2$, then (uw, t_3) is a temporal edge with $t_2 \leq t_3$.

[Mertzios, Molter, Renken, Spirakis, Zschoche, *MFCS*, 2021]

Temporal Transitivity



Temporal Transitivity

If (uv, t_1) and (vw, t_2) are temporal edges with $t_1 \leq t_2$, then (uw, t_3) is a temporal edge with $t_2 \leq t_3$.

Exchanging $\leq$ by $<$ yields four variants:
Temporal $(\{<, \leq\}, \{<, \leq\})$-**Transitivity**

- first "$<$" is called "strict"; second "$<$" is called "strong"

[Mertzios, Molter, Renken, Spirakis, Zschoche, *MFCS*, 2021]

Static Transitivity

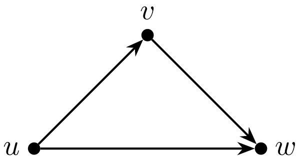
Definition

A graph is **transitively orientable** if its edges can be oriented such that, if uv and vw are oriented edges, then uw exists in the graph and is an oriented edge.

Static Transitivity

Definition

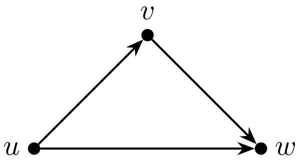
A graph is **transitively orientable** if its edges can be oriented such that, if uv and vw are oriented edges, then uw exists in the graph and is an oriented edge.

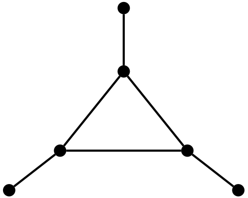


Static Transitivity

Definition

A graph is **transitively orientable** if its edges can be oriented such that, if uv and vw are oriented edges, then uw exists in the graph and is an oriented edge.



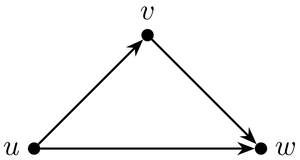


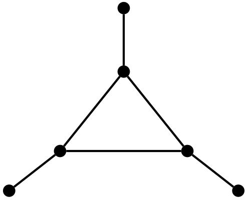
Forbidden induced subgraph

Static Transitivity

Definition

A graph is **transitively orientable** if its edges can be oriented such that, if uv and vw are oriented edges, then uw exists in the graph and is an oriented edge.





Forbidden induced subgraph

Transitively orientable graphs can be recognized in polynomial time [see.g. Golumbic '80].

Recognizing Temporal Transitivity I

- We assume for simplicity exactly one temporal label per edge

Temporal $(\leq, \leq)$-Transitivity

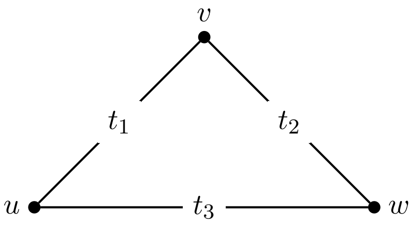
If (uv, t_1) and (vw, t_2) with $t_1 \leq t_2$, then (uw, t_3) with $t_2 \leq t_3$.

Recognizing Temporal Transitivity I

- We assume for simplicity exactly one temporal label per edge

Temporal $(\leq, \leq)$-Transitivity

If (uv, t_1) and (vw, t_2) with $t_1 \leq t_2$, then (uw, t_3) with $t_2 \leq t_3$.

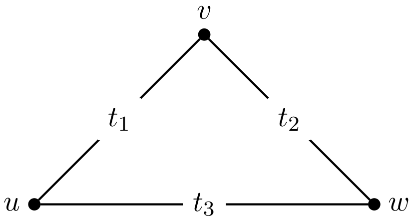


Recognizing Temporal Transitivity I

- We assume for simplicity exactly one temporal label per edge

Temporal $(\leq, \leq)$-Transitivity

If (uv, t_1) and (vw, t_2) with $t_1 \leq t_2$, then (uw, t_3) with $t_2 \leq t_3$.



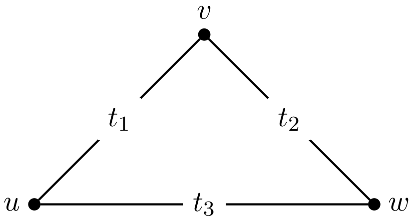
$$t_1 = t_2 = t_3 \;\big|\; t_1 < t_2 = t_3 \;\big|\; t_1 = t_2 < t_3 \;\big|\; t_1 < t_2 < t_3$$

Recognizing Temporal Transitivity I

- We assume for simplicity exactly one temporal label per edge

Temporal $(\leq, \leq)$-Transitivity

If (uv, t_1) and (vw, t_2) with $t_1 \leq t_2$, then (uw, t_3) with $t_2 \leq t_3$.



$t_1 = t_2 = t_3$ | $t_1 < t_2 = t_3$ | $t_1 = t_2 < t_3$ | $t_1 < t_2 < t_3$

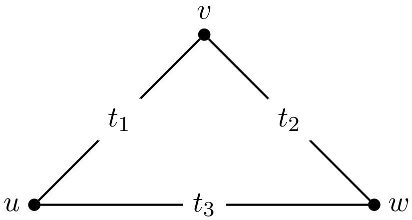
non-cyclic

Recognizing Temporal Transitivity I

- We assume for simplicity exactly one temporal label per edge

Temporal $(\le, \le)$-Transitivity

If (uv, t_1) and (vw, t_2) with $t_1 \le t_2$, then (uw, t_3) with $t_2 \le t_3$.



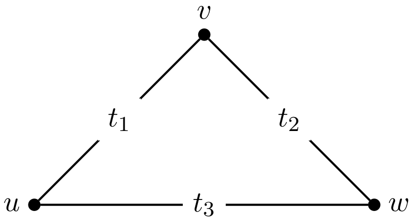
$t_1 = t_2 = t_3$	$t_1 < t_2 = t_3$	$t_1 = t_2 < t_3$	$t_1 < t_2 < t_3$
non-cyclic	$wu = wv$		

Recognizing Temporal Transitivity I

- We assume for simplicity exactly one temporal label per edge

Temporal $(\leq, \leq)$-Transitivity

If (uv, t_1) and (vw, t_2) with $t_1 \leq t_2$, then (uw, t_3) with $t_2 \leq t_3$.



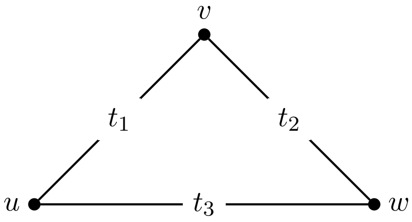
$t_1 = t_2 = t_3$	$t_1 < t_2 = t_3$	$t_1 = t_2 < t_3$	$t_1 < t_2 < t_3$
non-cyclic	$wu = wv$	$vw \implies uw$ $vu \implies wu$	

Recognizing Temporal Transitivity I

- We assume for simplicity exactly one temporal label per edge

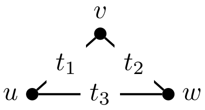
Temporal $(\leq, \leq)$-Transitivity

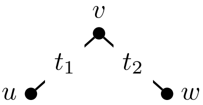
If (uv, t_1) and (vw, t_2) with $t_1 \leq t_2$, then (uw, t_3) with $t_2 \leq t_3$.



$t_1 = t_2 = t_3$	$t_1 < t_2 = t_3$	$t_1 = t_2 < t_3$	$t_1 < t_2 < t_3$
non-cyclic	$wu = wv$	$vw \implies uw$ $vu \implies wu$	$vw \implies uw$ $vu \implies wu$

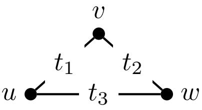
Recognizing Temporal Transitivity II

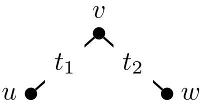




	$t_1 = t_2 = t_3$	$t_1 < t_2 = t_3$	$t_1 \leq t_2 < t_3$	$t_1 = t_2$	$t_1 < t_2$
$(\leq, \leq)$	non-cyclic	$wu = wv$	$vw \implies uw$ $vu \implies wu$	$uv = wv$	$uv \implies wv$

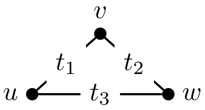
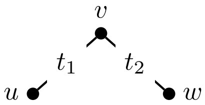
Recognizing Temporal Transitivity II



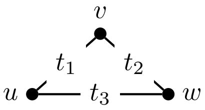


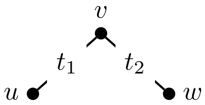
	$t_1 = t_2 = t_3$	$t_1 < t_2 = t_3$	$t_1 \leq t_2 < t_3$	$t_1 = t_2$	$t_1 < t_2$
$(\leq, \leq)$	non-cyclic	$wu = wv$	$vw \implies uw$ $vu \implies wu$	$uv = wv$	$uv \implies wv$
$(\leq, <)$	$\perp$	$wu \wedge wv$	$vw \implies uw$ $vu \implies wu$	$uv = wv$	$uv \implies wv$

Recognizing Temporal Transitivity II

					
	$t_1 = t_2 = t_3$	$t_1 < t_2 = t_3$	$t_1 \leq t_2 < t_3$	$t_1 = t_2$	$t_1 < t_2$
$(\leq, \leq)$	non-cyclic	$wu = wv$	$vw \implies uw$ $vu \implies wu$	$uv = wv$	$uv \implies wv$
$(\leq, <)$	$\perp$	$wu \wedge wv$	$vw \implies uw$ $vu \implies wu$	$uv = wv$	$uv \implies wv$
$(<, \leq)$	$\top$	non-cyclic	$vw \implies uw$ $vu \implies wu$	$\top$	$uv \implies wv$

Recognizing Temporal Transitivity II





	$t_1 = t_2 = t_3$	$t_1 < t_2 = t_3$	$t_1 \le t_2 < t_3$	$t_1 = t_2$	$t_1 < t_2$
$(\le, \le)$	non-cyclic	$wu = wv$	$vw \implies uw$ $vu \implies wu$	$uv = wv$	$uv \implies wv$
$(\le, <)$	$\bot$	$wu \wedge wv$	$vw \implies uw$ $vu \implies wu$	$uv = wv$	$uv \implies wv$
$(<, \le)$	$\top$	non-cyclic	$vw \implies uw$ $vu \implies wu$	$\top$	$uv \implies wv$
$(<, <)$	$\top$	$wu \wedge wv$	$vw \implies uw$ $vu \implies wu$	$\top$	$uv \implies wv$

Our Results

Recognizing Temporal Transitivity

- Recognizing of Non-Strict Temporal $(\leq, \leq)$-Transitivity in poly-time.
- Recognizing Strict Temporal $(<, \leq)$-Transitivity is NP-hard.
- Remaining ("strong") variants can be recognized in polynomial time.

Temporal Transitivity Completion
(given a partially oriented graph, add $\leq k$ edges, and one label per edge)

- All four variants are NP-hard.
- Poly-time if input graph is fully oriented.
- FPT wrt. number of unoriented edges in input graph.

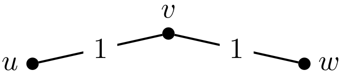
Recognizing Multilayer Transitivity
(permanent orientation of edges in a temporal graph)

- NP-hard.

[Mertzios, Molter, Renken, Spirakis, Zschoche, *MFCS*, 2021]

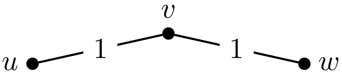
Poly-time Algorithm for Temporal $(\leq, \leq)$-Transitivity I

Important concept: **Forcing**:



Poly-time Algorithm for Temporal $(\leq, \leq)$-Transitivity I

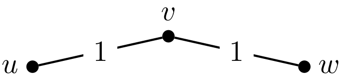
Important concept: **Forcing**:



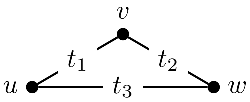
Main idea: Create a mixed Boolean formula $\phi_{\text{3NAE}} \wedge \phi_{\text{2SAT}}$ from:

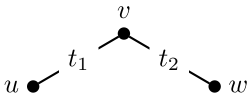
Poly-time Algorithm for Temporal $(\leq, \leq)$-Transitivity I

Important concept: **Forcing**:



Main idea: Create a mixed Boolean formula $\color{red}{\phi_{\text{3NAE}}} \wedge \color{blue}{\phi_{\text{2SAT}}}$ from:

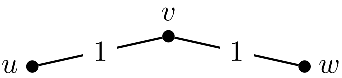




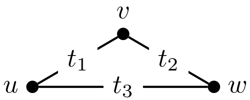
$t_1 = t_2 = t_3$	$t_1 < t_2 = t_3$	$t_1 \leq t_2 < t_3$	$t_1 = t_2$	$t_1 < t_2$
non-cyclic	$wu = wv$	$vw \implies uw$ $vu \implies wu$	$uv = wv$	$uv \implies wv$

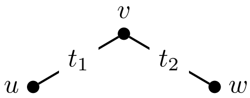
Poly-time Algorithm for Temporal $(\leq,\leq)$-Transitivity I

Important concept: **Forcing**:



Main idea: Create a mixed Boolean formula $\phi_{\text{3NAE}} \wedge \phi_{\text{2SAT}}$ from:





$t_1 = t_2 = t_3$	$t_1 < t_2 = t_3$	$t_1 \leq t_2 < t_3$	$t_1 = t_2$	$t_1 < t_2$
non-cyclic	$wu = wv$	$vw \implies uw$ $vu \implies wu$	$uv = wv$	$uv \implies wv$

Similar algorithm with solving 2SAT:

1. Set a variable, apply all "static forcings": if no contradiction, keep it.
2. Iteratively set truth values and replace ϕ_{3NAE}-clauses with ϕ_{2SAT}-clauses.

Poly-time Algorithm for Temporal $(\leq, \leq)$-Transitivity II

Some key insights:

Lemma

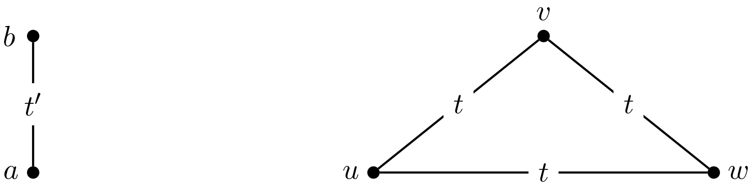
If orienting an edge forces orienting an edge in a "synchronous triangle", it also forces orienting a **different** edge in the **same** synchronous triangle.

Poly-time Algorithm for Temporal $(\leq, \leq)$-Transitivity II

Some key insights:

Lemma

If orienting an edge forces orienting an edge in a "synchronous triangle", it also forces orienting a **different** edge in the **same** synchronous triangle.

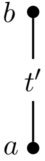


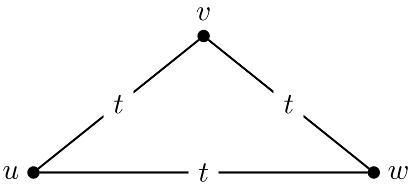
Poly-time Algorithm for Temporal $(\leq, \leq)$-Transitivity II

Some key insights:

Lemma

If orienting an edge forces orienting an edge in a "synchronous triangle", it also forces orienting a **different** edge in the **same** synchronous triangle.



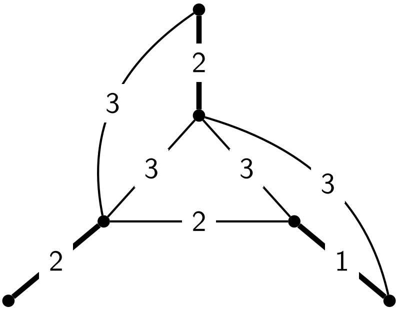


Lemma

Transforming NAE-clauses into 2SAT-clauses creates **no** "new implication chains".

NP-hardness of (Strict) Temporal $(<, \leq)$-Transitivity

Reduction from 3SAT. Clause gadget:



Observation

Not all three thick edges can be oriented inwards, two inwards and one outwards possible.

Overview

- Temporal graphs
- Temporal parameters and temporal paths: a warm-up
- Temporal vertex cover
- Temporal transitive orientations
- Stochastic temporal graphs

Stochastic Temporal Graphs

Levels of knowledge about the network evolution:

- whole temporal graph given in advance
- adversary who reveals it snapshot-by-snapshot at every time step
- intermediate knowledge setting, captured by **stochastic temporal graphs**, where the network evolution is given by a probability distribution that governs the appearance of each edge over time

[Akrida, Mertzios, Nikoletseas, Raptopoulos, Spirakis, Zamaraev, *J. Computer and System Sciences*, 2020]

Stochastic Temporal Graphs

Levels of knowledge about the network evolution:

- whole temporal graph given in advance
- adversary who reveals it snapshot-by-snapshot at every time step
- intermediate knowledge setting, captured by **stochastic temporal graphs**, where the network evolution is given by a probability distribution that governs the appearance of each edge over time

"Memory effect": appearance probability of a particular edge at a given time step t depends on the appearance (or absence) of the same edge at the previous $k \geq 1$ time steps

- faulty network communication

[Akrida, Mertzios, Nikoletseas, Raptopoulos, Spirakis, Zamaraev, *J. Computer and System Sciences*, 2020]

Stochastic Temporal Graphs

Memoryless case, $\mathcal{G}^{(0)}$:
$\forall e \in E$, $\forall t \in \mathbb{N}$, e appears in G_t with probability p_e.
The numbers $\{p_e : e \in E\}$ are given parameters of the model.

Stochastic Temporal Graphs

Memoryless case, $\mathcal{G}^{(0)}$:
$\forall e \in E$, $\forall t \in \mathbb{N}$, e appears in G_t with probability p_e.
The numbers $\{p_e : e \in E\}$ are given parameters of the model.

Memory-1, $\mathcal{G}^{(1)}$:
Initial snapshot $G_0 \subseteq G$.

$\forall e \in E$, $\forall t \in \mathbb{N}$:

- if e was absent in G_{t-1}, e appears in G_t with probability p_e and is absent with probability $1 - p_e$
- if e appeared in G_{t-1}, e appears in G_t with probability $1 - q_e$ and is absent with probability q_e

$$M_e = \left(\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1 - p_e & p_e \\ 1 & q_e & 1 - q_e \end{array} \right), \text{ where } 0 \leq p_e, q_e \leq 1.$$

Stochastic Temporal Graphs

Memoryless case, $\mathcal{G}^{(0)}$:
$\forall e \in E$, $\forall t \in \mathbb{N}$, e appears in G_t with probability p_e.
The numbers $\{p_e : e \in E\}$ are given parameters of the model.

Memory-1, $\mathcal{G}^{(1)}$:
Initial snapshot $G_0 \subseteq G$.

$\forall e \in E$, $\forall t \in \mathbb{N}$:

- if e was absent in G_{t-1}, e appears in G_t with probability p_e and is absent with probability $1 - p_e$
- if e appeared in G_{t-1}, e appears in G_t with probability $1 - q_e$ and is absent with probability q_e

$$M_e = \left(\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1-p_e & p_e \\ 1 & q_e & 1-q_e \end{array}\right), \text{ where } 0 \leq p_e, q_e \leq 1.$$

If $p_e = p$ and $q_e = q$, $\forall e$, we have exactly the *edge-Markovian evolving graph* model introduced by Clementi et al. (SIAM Journal on Discrete Mathematics '10).

Stochastic Temporal Graphs

Memoryless case, $\mathcal{G}^{(0)}$:
$\forall e \in E$, $\forall t \in \mathbb{N}$, e appears in G_t with probability p_e.
The numbers $\{p_e : e \in E\}$ are given parameters of the model.

Memory-k, $\mathcal{G}^{(k)}$:
Initial sequence of k snapshots $G_{-k+1}, \ldots, G_{-1}, G_0 \subseteq G$.

$\forall e \in E$, $\forall t \in \mathbb{N}$:

- e appears with probability $p_e(H_e^{(k)})$ that depends only on the **history** $H_e^{(k)}$ **of its appearance in the last k snapshots**.
- at every time step t, this history is a k-bit binary vector, where a 0-entry (resp. 1-entry) on the i-th position denotes absence (resp. appearance) of e in $E_{t-k+i-1}$, for $i = 1, \ldots, k$

Stochastic Temporal Graphs

Memoryless case, $\mathcal{G}^{(0)}$:
$\forall e \in E$, $\forall t \in \mathbb{N}$, e appears in G_t with probability p_e.
The numbers $\{p_e : e \in E\}$ are given parameters of the model.

Memory-k, $\mathcal{G}^{(k)}$:
Initial sequence of k snapshots $G_{-k+1}, \ldots, G_{-1}, G_0 \subseteq G$.

$\forall e \in E$, $\forall t \in \mathbb{N}$:

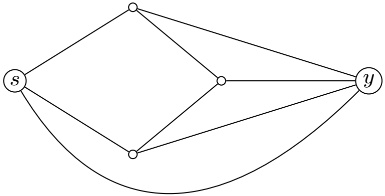
- e appears with probability $p_e(H_e^{(k)})$ that depends only on the **history** $H_e^{(k)}$ **of its appearance in the last k snapshots**.
- at every time step t, this history is a k-bit binary vector, where a 0-entry (resp. 1-entry) on the i-th position denotes absence (resp. appearance) of e in $E_{t-k+i-1}$, for $i = 1, \ldots, k$

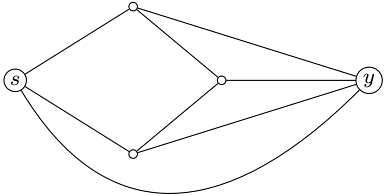
For every $k \geq 1$, the memory-$(k-1)$ model is a special case of the memory-k model.

The problems

Unbounded number of messages: "Flooding" the network with information

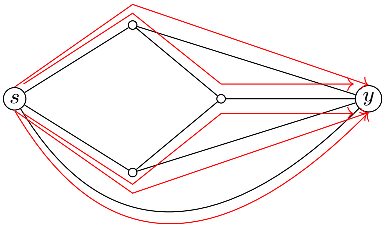
Limited number of messages: transferring a package with a tangible good



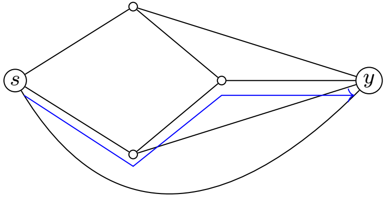


The problems

Unbounded number of messages: "Flooding" the network with information



Limited number of messages: transferring a package with a tangible good

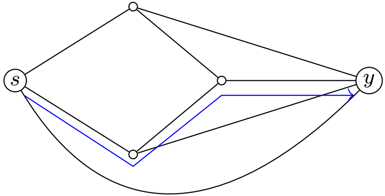


The problems

MINIMUM ARRIVAL:

Given a stochastic temporal graph on an underlying graph $G = (V, E)$ and two distinct vertices $s, y \in V$, compute the **expected arrival time of a foremost s-y journey**, $\mathbb{E}[X(s, y)]$.

Limited number of messages: transferring a package with a tangible good



The problems

Minimum Arrival:

Given a stochastic temporal graph on an underlying graph $G = (V, E)$ and two distinct vertices $s, y \in V$, compute the **expected arrival time of a foremost** s-y **journey**, $\mathbb{E}[X(s, y)]$.

Best Policy:

Every day t Alice "wakes up" in the morning located at vertex s_t and looks at which edges are available in today's snapshot; by only knowing her current position, the history of the last k snapshots, and the probabilistic rules of edge appearance, Alice needs to decide whether:

- to stay at the vertex s_t she currently is, or
- to use an edge of G_t to move to a neighbouring vertex.

The problems

MINIMUM ARRIVAL:

Given a stochastic temporal graph on an underlying graph $G = (V, E)$ and two distinct vertices $s, y \in V$, compute the **expected arrival time of a foremost** s-y **journey**, $\mathbb{E}[X(s, y)]$.

BEST POLICY:

Given a stochastic temporal graph on an underlying graph $G = (V, E)$ and two distinct vertices $s, y \in V$, compute the **expected arrival time of a best policy** s-y **journey**, $\mathbb{E}[Y(s, y)]$.

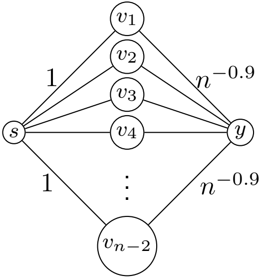
The problems

MINIMUM ARRIVAL:

Arrival time of the foremost journey from s to y will be equal to the first day after day 1 on which some edge incident to y appears.

Time needed for that follows geometric distribution, with success probability $1 - \left(1 - n^{-0.9}\right)^{n-2} = 1 - o(1)$.

So, solution is: $E[X(s,y)] = 2 + o(1)$.



BEST POLICY:

Any best policy for Alice will cross an edge incident to s on day 1 and then wait until the "next" edge in the path, incident to y, appears.

Time needed for that to happen is $n^{0.9}$.

So, solution is: $E[Y(s,y)] = 1 + n^{0.9}$.

Our results

MINIMUM ARRIVAL:

- #P-hard (even for the memoryless case)
- Approximation Scheme for memory-0 on series-parallel graphs
- Fully Polynomial Randomized Approximation Scheme (FPRAS) for memory-k, $k \geq 0$

BEST POLICY:

- #P-hard for memory-k, $k \geq 3$
- Formulation as MDP, leading to exact doubly-exponential-time algorithm
- Polynomial-time dynamic programming algorithm for the memoryless case
  - Studied before; different approaches; polynomial-time solutions e.g. Ogier and Rutenburg, Infocom '92 & Basu et al., arXiv

Research Directions

- Parameterized versions of the problems (with the appropriate parameters)
- Approximation algorithms
- Special temporal graph classes
  - e.g. the class of temporally orientable temporal graphs...?
- Distinction for path problems: strict vs. non-strict
  - the same distinction also on derived notions, e.g. temporal transitivity
- Other meaningful temporal graph problems
  - lifting "algorithmic graph theory" to the temporal case
- Need for experimental algorithms
  - Experimental Algorithms are needed especially for the provably hard problems here

Thank you for your attention!